Affine Hadamard families stemming from Fourier matrices

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Abstract

Complex Hadamard matrices are complex $N \times N$ matrices H such that:

- entries of H are unimodular, $|H_{i,j}| = 1$
- rows (as well as columns) of H are orthogonal, $H^*H = HH^* = N \cdot I_N$

An affine Hadamard family stemming from a complex $N \times N$ Hadamard matrix H is a set of the form:

$$\{H \circ \mathbf{EXP}(\mathbf{i}R) \; : \quad R \in \mathcal{R} \;\; \text{a subspace of all real } N \times N \text{ matrices}\} \tag{1}$$

where $\circ$ denotes the Hadamard product, and $[\mathbf{EXP}(\mathbf{i}R)]_{i,j} = e^{\mathbf{i}R_{i,j}}$, $i, j = 1..N$.

A method for describing and classifying all maximal affine Hadamard families stemming from Fourier matrices ($[F_N]_{i,j} = e^{\mathbf{i}2\pi(i-1)(j-1)/N}$) is presented.

The method uses so called *trees of division* which describe how the set:

$$N\text{–}box \overset{def}{=} \underbrace{Z_{a_1} \times .. \times Z_{a_1}}_{b_1 \text{ times}} \times \ldots \times \underbrace{Z_{a_q} \times .. \times Z_{a_q}}_{b_q \text{ times}}, \tag{2}$$

where N decomposes into primes $N = a_1^{b_1} \ldots a_q^{b_q}$, is split into 'subboxes', not necessarily of the same dimension.

Each tree of division corresponds to a system of linear equations involving the real entries $R_{i,j}$ of R. A whole set of appropriately constructed trees of division brings together such systems to form a huge system defining $\mathcal{R}$ in (1), thus designating one of the maximal affine Hadamard families stemming from $H = F_N$.

The families can be described in terms of trees of division thanks to the fact that the cyclic group of the N-th roots of unity and the product of cyclic groups $Z_{a_1^{b_1}} \times ... \times Z_{a_q^{b_q}}$ are isomorphic.

Keywords

Fourier matrices, Hadamard matrices, Complex Hadamard matrices, Linear systems.